Relation between Interfacial Separation and Load: A General Theory of Contact Mechanics

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I study the contact between a rigid solid with a randomly rough surface and an elastic block with a flat surface. I derive a relation between the (average) interfacial separation u and the applied normal squeezing pressure p. I show that, for nonadhesive interaction and small applied pressure, $p \sim \exp(-u/u_0)$, in good agreement with recent experimental observations.

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When two elastic solids with rough surfaces are squeezed together, the solids will, in general, make contact not everywhere in the apparent contact area but only at a distribution of asperity contact spots [1–4]. The separation $u(\mathbf{x})$ between the surfaces will vary in a nearly random way with the lateral coordinates $\mathbf{x} = (x, y)$ in the apparent contact area. When the applied squeezing pressure increases, the average surface separation $u = \langle u(\mathbf{x}) \rangle$ will decrease. In most situations, however, it is impossible to squeeze the solids into perfect contact corresponding to $u = 0$. The space between two solids has a tremendous influence on many important processes, e.g., heat transfer [5], contact resistivity [6], lubrication [7], sealing [8], optical interference [9], etc. In this Letter, I will present a very simple theory for the (average) separation u as a function of the squeezing pressure p. I will show that, for randomly rough surfaces at low squeezing pressures, $p \sim \exp(-u/u_0)$, where the reference length u_0 depends on the nature of the surface roughness but is independent of p, which is in good agreement with experiments [9].

We consider the frictionless contact between elastic solids with randomly rough surfaces. If $z = h_1(\mathbf{x})$ and $h_2(\mathbf{x})$ describe the surface profiles, E_1 and E_2 are Young's elastic moduli of the two solids, and ν_1 and ν_2 are the corresponding Poisson ratios, then the elastic contact problem is equivalent to the contact between a rigid solid (substrate) with the roughness profile $h(\mathbf{x}) = h_1(\mathbf{x}) + h_2(\mathbf{x})$, in contact with an elastic solid (block) with a flat surface and with Young's modulus E and Poisson ratio ν chosen so that [10]

$$\frac{1-\nu^2}{E} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}.$$

Introduce a coordinate system xyz with the xy plane in the average surface plane of the rough substrate and the z axis pointing away from the substrate; see Fig. 1. The separation between the average surface plane of the block and the average surface plane of the substrate is denoted by u, with $u \geq 0$. When the applied squeezing force p increases, the separation between the surfaces at the interface will decrease, so we can consider $p = p(u)$ as a function of u. The elastic energy $U_{\rm el}(u)$ stored in the substrate asperity-elastic block contact regions must equal the work done by the external pressure p in displacing the lower surface of the block towards the substrate, i.e.,

$$\int_u^\infty du' A_0 p(u') = U_{\rm el}(u) \tag{1}$$

or

$$p(u) = -\frac{1}{A_0}\frac{dU_{\rm el}}{du}, \tag{2}$$

where A_0 is the nominal contact area. Equation (2) is exact and shows that, if the dependence of the surface separation u on the squeezing pressure p is known, e.g., from finite element calculations or molecular dynamics, one can obtain the elastic energy $U_{\rm el}$ stored in the asperity contact regions [11]. This is an important result as $U_{\rm el}(u)$ is relevant for many important applications.

Theories show that, for low squeezing pressure, the area of real contact A varies linearly with the squeezing force pA_0 and that the interfacial stress distribution and the size

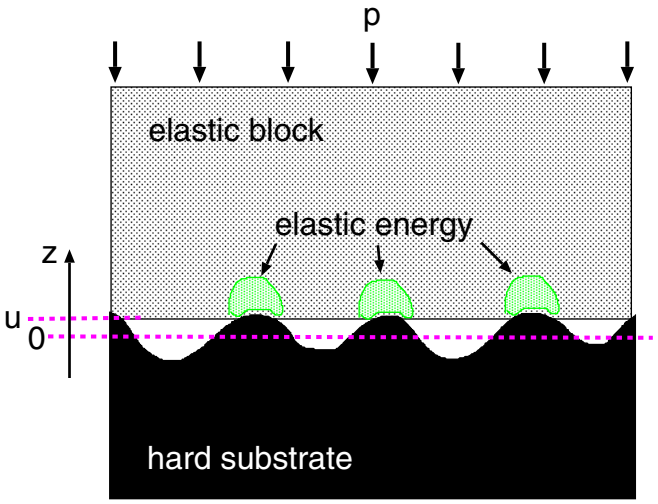

FIG. 1 (color online). An elastic block squeezed against a rigid rough substrate. The separation between the average plane of the substrate and the average plane of the lower surface of the block is denoted by u. Elastic energy is stored in the block in the vicinity of the asperity contact regions.

distribution of contact spots are independent of the squeezing pressure [12,13]. That is, with increasing p, existing contact areas grow and new contact areas form in such a way that, in the thermodynamic limit (infinite-sized system), the quantities referred to above remain unchanged. It follows immediately that for a small load the elastic energy stored in the asperity contact region will increase linearly with the load, i.e., $U_{\rm el}(u) = u_0 A_0 p(u)$, where u_0 is a characteristic length which depends on the surface roughness (see below) but is independent of the squeezing pressure p. Thus, for small pressures, (2) takes the form

$$p(u) = -u_0 \frac{dp}{du}$$

or

$$p(u) \sim e^{-u/u_0} \tag{3}$$

in good agreement with experimental data for the contact between elastic solids when the adhesional interaction between the solids can be neglected [9]. We note that the result (3) differs drastically from the prediction of the theory of Bush, Gibson, and Thomas [14] and that of Greenwood and Williamson (GW) [15], which for low squeezing pressures (for randomly rough surfaces with Gaussian height distribution) predict $p(u) \sim u^{-a} \exp(-bu^2)$, where $a = 1$ in the theory of Bush, Gibson, and Thomas and $a = 5/2$ in the GW theory. Thus, these theories do not correctly describe the interfacial spacing between contacting solids. This is not surprising, because these approaches assume a rigid substrate surface covered with flexible asperities. In reality, the bulk of the solids whose surfaces are in contact is not rigid. Furthermore, there exists a hierarchy of asperities on many length scales, all of which can distort.

The elastic energy $U_{\rm el}$ in the simplest approximation takes the form [11,13,16,17]

$$U_{\rm el} \approx A_0 E^* \frac{\pi}{2} \gamma \int_{q_0}^{q_1} dq q^2 P(q) C(q), \tag{4}$$

where $E^* = E/(1-\nu^2)$ and where $P(q) = A(\zeta)/A_0$ is the relative contact area when the interface is studied at the magnification $\zeta = q/q_0$. The surface roughness power spectrum [18]

$$C(q) = \frac{1}{(2\pi)^2} \int d^2x \langle h(\mathbf{x}h(\mathbf{0})\rangle e^{-i\mathbf{q}\cdot\mathbf{x}},$$

where $\langle \cdots \rangle$ stands for ensemble average. For high squeezing pressures, perfect contact occurs at the interface, and in this case $P = 1$ and with $\gamma = 1$, Eq. (4) is exact. Below, we will focus mainly on low squeezing pressures and for this case $\gamma < 1$ (but of order unity) to take into account that the elastic energy stored in the contact region (per unit surface area) is less than the average elastic energy (per unit surface area) for perfect contact [11]. We will use the contact mechanics theory of Persson, where for elastic nonadhesive contact the function [19,20]

$$P(q) = \frac{2}{\sqrt{\pi}} \int_0^{s(q)p} dx e^{-x^2}, \tag{5}$$

where $s(q) = w(q)/E^*$, with

$$w(q) = \left(\pi \int_{q_0}^{q} dq' q'^3 C(q') \right)^{-1/2}. \tag{6}$$

Using (5) gives

$$\frac{\partial P}{\partial u} = \frac{2}{\sqrt{\pi}} s \frac{dp}{du} e^{-s^2 p^2}. \tag{7}$$

Substituting (4) and (7) in (2) gives

$$p(u) = -\sqrt{\pi}\gamma \int_{q_0}^{q_1} dq q^2 C(q) w(q) e^{-[w(q)p/E^*]^2} \frac{dp}{du}$$

or

$$du = -\sqrt{\pi}\gamma \int_{q_0}^{q_1} dq q^2 C(q) w(q) e^{-[w(q)p/E^*]^2} \frac{dp}{p}.$$

Integrating this from $u = 0$ (complete contact, corresponding to $p = \infty$) to u gives

$$u = \sqrt{\pi}\gamma \int_{q_0}^{q_1} dq q^2 C(q) w(q) \int_p^{\infty} dp' \frac{1}{p'} e^{-[w(q)p'/E^*]^2}. \tag{8}$$

For very small pressures, we get from (8):

$$u = -u_0 \log(p/p_c)$$

or

$$p = p_c e^{-u/u_0}, \tag{9}$$

where

$$u_0 = \sqrt{\pi}\gamma \int_{q_0}^{q_1} dq q^2 C(q) w(q) \tag{10}$$

and where the cutoff p_c is determined by

$$p_c = \epsilon E^* \exp(-\langle \log w \rangle), \tag{11}$$

with

$$\epsilon = \exp\left(\int_0^{\infty} dx 2x \log x e^{-x^2} \right) \approx 0.7493$$

and where

$$\langle \log w \rangle = \frac{\int_{q_0}^{q_1} dq q^2 C(q) w(q) \log w(q)}{\int_{q_0}^{q_1} dq q^2 C(q) w(q)}. \tag{12}$$

If we assume that the substrate surface roughness is self-affine fractal for $q_0 < q < q_1$, we get [18]

$$C(q) = \frac{H}{\pi} \frac{\langle h^2 \rangle}{q_0^2} \left(\frac{q}{q_0} \right)^{-2(H+1)}, \tag{13}$$

where $H = 3 - D_f$, where D_f is the fractal dimension.

The mean of the square of the substrate surface height profile is $\langle h^2 \rangle = h_{\rm rms}^2$. Substituting (13) in (10) gives

$$u_0 = h_{\rm rms}/\alpha, \tag{14}$$

where

$$\alpha^{-1} = \left(\frac{2H(1-H)}{\pi}\right)^{1/2} \gamma \int_1^{q_1/q_0} dx g(x), \tag{15}$$

with

$$g(x) = x^{-H}(x^2 - x^{2H})^{-1/2}.$$

Substituting (13) in (12) gives

$$\langle \log w \rangle = -\log(q_0 h_{\rm rms}) - \log\beta, \tag{16}$$

where

$$\log\beta = \frac{\int_1^{q_1/q_0} dx g(x) \log[\frac{H}{2(1-H)}(x^{2(1-H)} - 1)]}{2\int_1^{q_1/q_0} dx g(x)}. \tag{17}$$

Substituting (14), (16), and (11) in (9) gives

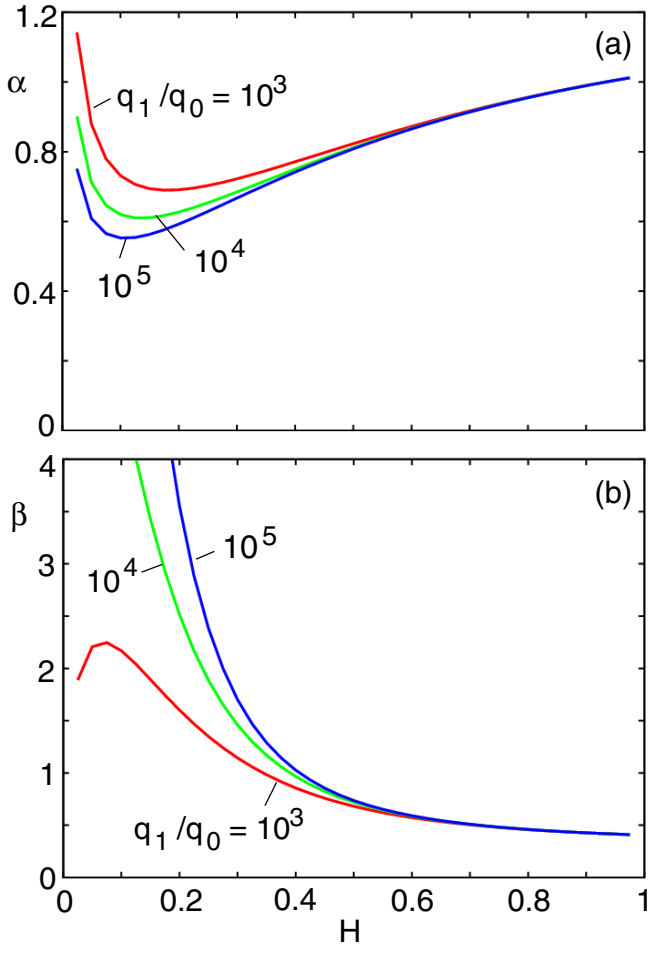

FIG. 2 (color online). The parameters α and β as a function of the Hurst exponent H for three different values of the ratio q_1/q_0.

$$p = \beta\epsilon q_0 h_{\rm rms} E^* e^{-\alpha u/h_{\rm rms}} \tag{18}$$

or

$$u = \alpha^{-1} h_{\rm rms} \log(\beta\epsilon q_0 h_{\rm rms} E^*/p). \tag{19}$$

In Fig. 2, we show the parameters α (for $\gamma = 1$) and β as a function of the Hurst exponent H for three different values of the ratio q_1/q_0. Most surfaces which are self-affine fractal have the Hurst exponent $H > 0.5$ (or the fractal dimension $D_f < 2.5$). For such surfaces, the parameters α and β are nearly independent of the ratio q_1/q_0 between the highest q_1 and smallest q_0 wave vector included in the analysis. Asphalt road surfaces [18] or surfaces prepared by crack propagation [21] or by bombardment with particles typically have $H \approx 0.8$ (or the fractal dimension $D_f = 3 - H \approx 2.2$), for which case $\alpha \approx 1$ and $\beta \approx 0.5$. In this case, we have

$$p \approx 0.5\epsilon q_0 h_{\rm rms} E^* e^{-u/\gamma h_{\rm rms}} \tag{20}$$

or

$$u \approx \gamma h_{\rm rms} \log(0.5\epsilon q_0 h_{\rm rms} E^*/p), \tag{21}$$

where we have reintroduced the factor of γ.

In Fig. 3, we show the relation (for $\gamma = 1$) between the (natural) logarithm of the squeezing pressure p and the interfacial separation u for an elastic solid squeezed against a rigid, self-affine fractal surface with the Hurst exponent $H = 0.8$, as calculated directly from (8) by numerical integration. The dashed line shows the large-distance asymptotic behavior given by (21). Note that deviation from this logarithmic relation occurs only for $u < h_{\rm rms}$ and that there is a sharp increase in the squeezing

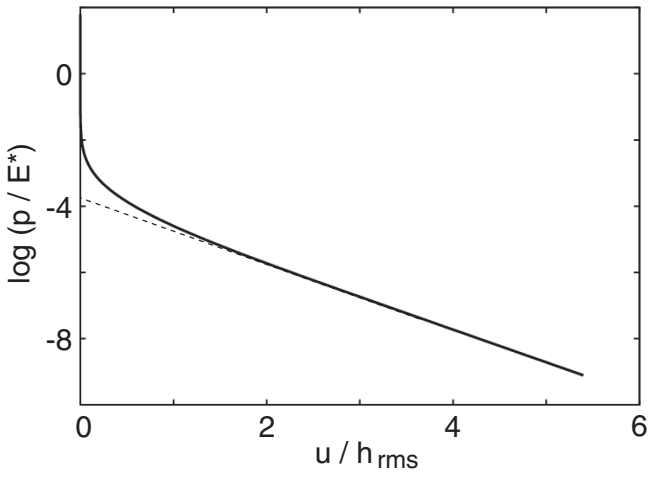

FIG. 3. The relation between the (natural) logarithm of the squeezing pressure p and the interfacial separation u for an elastic solid squeezed against a rigid, self-affine fractal surface with the Hurst exponent $H = 0.8$. The surface has the rms roughness $h_{\rm rms} = 6\ \mu$m, and the upper and lower cutoff wave vectors are $q_1 = 7.8 \times 10^9\ {\rm m}^{-1}$ and $q_0 = 1 \times 10^4\ {\rm m}^{-1}$, respectively.

pressure for $u < h_{\rm rms}$. Both of these facts are in accordance with the experimental data presented in Ref. [9]. Note that the slope of the dashed line in Fig. 3 is close to -1, as expected from (21), and that the dashed curve for $u = 0$ gives $\log(p/E^*) \approx -3.8$ or $p/E^* \approx 0.02$, which is the same as the prediction of (21): $0.5\epsilon q_0 h_{\rm rms} \approx 0.02$. I have performed calculations of the $p(u)$ relation using several measured surface roughness power spectra $C(q)$, but the general form is always as in Fig. 3, and the slope of the line (using $\gamma = 1$) in the linear region is always close to -1.

Pei *et al.* [22] have performed a finite element computer simulation of the contact mechanics for a polymer surface, using the measured surface topography [9] as input, squeezed against a flat surface. They found that for a large separation $p \sim \exp(-u/\gamma h_{\rm rms})$, with $\gamma \approx 0.4$, which is consistent with our numerical and analytical results.

The theory presented above can be easily generalized in various ways. Thus, it is possible to include the adhesional interaction [23,24]. In this case, the work done by the external pressure p will be the sum of the stored (asperity-induced) elastic energy plus the (negative) adhesional energy; i.e., the right-hand side of (1) will now be $U_{\rm el} + U_{\rm ad}$. The theory can also be applied to study how the spacing $u(\zeta)$ depends on the magnification. Here $u(\zeta)$ is the (average) spacing between the solids in the apparent contact areas observed at the magnification ζ. It can be obtained from the equations above by replacing q_0 with $q_0\zeta$ and the nominal squeezing pressure p with $p(\zeta) = pA_0/A(\zeta)$. The quantity $u(\zeta)$ is of crucial importance for lubricated seals [25]. The results of these generalizations of the theory will be presented elsewhere.

Finally, we note that the observation of an effective exponential repulsion has important implications for tribology, colloid science, powder technology, and materials science [9]. For example, the density or volume of granular materials has long been known to have a logarithmic dependence on the externally applied isotropic pressure or stress, as found, for example, in the compression stage during processing of ceramic materials [26]. Recent work on the confinement of nanoparticles has also indicated an exponential force upon compression [27], suggesting that this relationship could be prevalent among quite different types of heterogeneous surfaces.

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