

## FAST TRACK COMMUNICATION

# On the elastic energy and stress correlation in the contact between elastic solids with randomly rough surfaces

B N J Persson

IFF, FZ-Jülich, 52425 Jülich, Germany

Received 3 June 2008, in final form 12 June 2008

Published 26 June 2008

Online at [stacks.iop.org/JPhysCM/20/312001](http://stacks.iop.org/JPhysCM/20/312001)**Abstract**

When two elastic solids with randomly rough surfaces are brought in contact, a very inhomogeneous stress distribution  $\sigma(\mathbf{x})$  will occur at the interface. Here I study the elastic energy and the correlation function  $\langle \sigma(\mathbf{q})\sigma(-\mathbf{q}) \rangle$ , where  $\sigma(\mathbf{q})$  is the Fourier transform of  $\sigma(\mathbf{x})$  and where  $\langle \dots \rangle$  stands for ensemble average. I relate  $\langle \sigma(\mathbf{q})\sigma(-\mathbf{q}) \rangle$  to the elastic energy stored at the interface, and I show that for self-affine fractal surfaces, quite generally  $\langle \sigma(\mathbf{q})\sigma(-\mathbf{q}) \rangle \sim q^{-(1+H)}$ , where  $H$  is the Hurst exponent of the self-affine fractal surface.

The contact between elastic solids with surfaces with roughness on many length scales is a topic of great practical importance. Particularly important are the area of real contact  $A$ , which in most cases determines the sliding friction force, and the asperity induced elastic energy  $U_{\text{el}}$  stored at the interface, which is directly related to the interfacial separation [1].

In a recent paper Campana *et al* [2] have presented numerical simulation results for the stress distribution at the interface between two elastic solids with randomly rough (self-affine fractal) surfaces. They have shown that the stress correlation function

$$\langle \sigma(\mathbf{q})\sigma(-\mathbf{q}) \rangle \sim q^{-(1+H)}$$

where  $H$  is the Hurst exponent of the self-affine fractal surface. Here I show that this is also the prediction of the contact mechanics theory of Persson [3]. I also derive some other results related to the elastic energy and the accuracy of the Persson theory.

Consider the contact between two elastic solids with rough surfaces, but which appear flat at low resolution. We can write the elastic energy stored in the vicinity of the asperity contact regions as

$$U_{\text{el}} = \frac{1}{2} \int d^2x \langle \sigma(\mathbf{x})u(\mathbf{x}) \rangle \quad (1)$$

where  $u(\mathbf{x})$  and  $\sigma(\mathbf{x})$  are the normal displacement and the

normal stress, respectively. We write

$$\sigma(\mathbf{x}) = \int d^2q \sigma(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}} \quad (2)$$

and similar for  $u(\mathbf{x})$ . Substituting this in (1) gives

$$U_{\text{el}} = (2\pi)^2 \frac{1}{2} \int d^2q \langle \sigma(\mathbf{q})u(-\mathbf{q}) \rangle. \quad (3)$$

Next, using that [3]

$$\sigma(\mathbf{q}) = \frac{1}{2} E^* q u(\mathbf{q}) \quad (4)$$

where  $E^* = E/(1-\nu^2)$  is the effective elastic modulus (where  $E$  is the Young's modulus and  $\nu$  the Poisson ratio) gives

$$U_{\text{el}} = (2\pi)^2 \frac{1}{E^*} \int d^2q q^{-1} \langle \sigma(\mathbf{q})\sigma(-\mathbf{q}) \rangle \quad (5)$$

and

$$U_{\text{el}} = (2\pi)^2 \frac{E^*}{4} \int d^2q q \langle u(\mathbf{q})u(-\mathbf{q}) \rangle. \quad (6)$$

For complete contact  $u(\mathbf{q}) = h(\mathbf{q})$  and using the definition

$$\langle h(\mathbf{q})h(-\mathbf{q}) \rangle = \frac{A_0}{(2\pi)^2} C(q) \quad (7)$$

gives for complete contact

$$U_{\text{el}} = \frac{E^* A_0}{4} \int d^2q q C(q). \quad (8)$$

Now, for incomplete contact I define  $W(q)$  so that

$$U_{el} = \frac{E^* A_0}{4} \int d^2q q C(q) W(q). \quad (9)$$

I have argued elsewhere [4] that  $W(q) = P(q) = A(\zeta)/A_0$  is the relative contact area when the interface is studied at the magnification  $\zeta = q/q_0$ . The qualitative explanation is that the solids will deform mainly in the regions where they make contact and most of the elastic energy will arise from the contact regions. In particular, for complete contact  $P(q) = 1$  and in this limit using  $W = P = 1$  would be exact. I will show below that  $W(q) = P(q)$  follows directly from the contact mechanics theory of Persson, but for the moment we simply define  $W(q)$  via (9).

Comparing (5) and (9) gives

$$\langle \sigma(\mathbf{q}) \sigma(-\mathbf{q}) \rangle = \left( \frac{E^*}{4\pi} \right)^2 A_0 q^2 C(q) W(q). \quad (10)$$

In what follows we need

$$\begin{aligned} & \int d^2q \langle \sigma(\mathbf{q}) \sigma(-\mathbf{q}) \rangle \\ &= \frac{1}{(2\pi)^4} \int d^2x d^2x' \int d^2q \langle \sigma(\mathbf{x}) \sigma(\mathbf{x}') \rangle e^{-i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \\ &= \frac{1}{(2\pi)^2} \int d^2x \langle \sigma^2(\mathbf{x}) \rangle = \frac{A_0}{(2\pi)^2} \langle \sigma^2 \rangle. \end{aligned} \quad (11)$$

Substituting (10) in (11) gives

$$\int_{q_0}^q dq q^3 C(q) W(q) = \frac{2}{\pi} \frac{\langle \sigma^2 \rangle}{E^{*2}}. \quad (12)$$

All the equations (1)–(12) presented above are exact.

Now, it is easy to calculate  $\langle \sigma^2 \rangle$  approximately using the Persson contact mechanics theory [3]. The basic equation for the stress distribution  $P(\sigma, \zeta)$

$$\frac{\partial P}{\partial \zeta} = f(\zeta) \frac{\partial^2 P}{\partial \sigma^2} \quad (13)$$

where

$$f(\zeta) = \frac{\pi}{4} E^{*2} q_0^3 C(q) \quad (14)$$

where  $q = q_0 \zeta$ . Multiply (13) with  $\sigma^2$  and integrate over  $\sigma$ :

$$\begin{aligned} \frac{d\langle \sigma^2 \rangle}{d\zeta} &= f(\zeta) \int_0^\infty \sigma^2 \frac{\partial^2 P}{\partial \sigma^2} \\ &= 2f(\zeta) \int_0^\infty d\sigma P(\sigma, \zeta) \end{aligned} \quad (15)$$

where I have performed two partial integrations. Now, note that

$$\int_0^\infty d\sigma P(\sigma, \zeta) = \frac{A(\zeta)}{A_0} = P(\zeta) \quad (16)$$

which I also denote as  $P(q)$  ( $q = q_0 \zeta$ ) for simplicity. Using the definition of  $f(\zeta)$  we get

$$\langle \sigma^2 \rangle = \frac{\pi}{2} E^{*2} \int_{q_0}^q dq q^3 C(q) P(q). \quad (17)$$

Substituting (17) in (12) gives

$$\int_{q_0}^q dq q^3 C(q) W(q) = \int_{q_0}^q dq q^3 C(q) P(q). \quad (18)$$

Thus we get  $W(q) = P(q)$ . The resulting equation for the elastic energy was used in [4], but only qualitative arguments were given for its validity (see above). Here we have proved that it follows rigorously within the Persson contact mechanics theory. Substituting this result in (10) gives

$$\langle \sigma(\mathbf{q}) \sigma(-\mathbf{q}) \rangle = \left( \frac{E^*}{4\pi} \right)^2 A_0 q^2 C(q) P(q). \quad (19)$$

Assume now a self-affine fractal surface. In this case

$$C(q) \sim q^{-2(H+1)}. \quad (20)$$

If we assume that the relative contact area is small then

$$P(q) \approx [\pi G(q)]^{-1/2} \quad (21)$$

where

$$\begin{aligned} G(q) &= \left( \frac{E^*}{\sigma_0} \right)^2 \frac{\pi}{4} \int_{q_0}^q dq q^3 C(q) \\ &\sim q^{2(1-H)} - q_0^{2(1-H)} \approx q^{2(1-H)} \end{aligned} \quad (22)$$

when  $q \gg q_0$ . Thus,

$$P(q) \sim q^{H-1} \quad (23)$$

when the contact is small and  $q \gg q_0$ . Substituting these results into (19) gives the  $q$ -dependence

$$\langle \sigma(\mathbf{q}) \sigma(-\mathbf{q}) \rangle \sim q^{-(1+H)} \quad (24)$$

in good agreement with recent numerical studies of contact between elastic solids with randomly rough surfaces [2]. It is interesting to note that contact mechanics models which neglect the long-ranged elastic deformation, such as the asperity contact models of Greenwood and Williamson [5] and of Bush *et al* [6], predict incorrect scaling where  $1 + H$  in (24) is replaced by  $2(1 + H)$  (see [2]). Thus the lateral correlation of the stress, as reflected by the correlation function

$$\langle \sigma(\mathbf{x}) \sigma(\mathbf{0}) \rangle = \frac{(2\pi)^2}{A_0} \int d^2q \langle \sigma(\mathbf{q}) \sigma(-\mathbf{q}) \rangle e^{i\mathbf{q} \cdot \mathbf{x}}$$

is incorrectly described by these theories.

Detailed analysis of molecular dynamics [7], finite element method [8] and Green's function molecular dynamics [9] calculations indicate some small deviations from the prediction of the theory of Persson. Thus, for self-affine fractal surfaces with the fractal dimension  $D_f \approx 2.2$  (which is typical for many surfaces of engineering interest) the contact area calculated as a function of the squeezing pressure tend to be about  $\sim 20\%$  larger at small pressures (where  $A$  varies linearly with  $p$ ) than predicted by the Persson theory. When the squeezing pressure increases, the deviation becomes smaller and vanishes at complete contact. Similarly, analysis of the (average)

interfacial separation as a function of the applied pressure (or load) indicate that the elastic energy in the asperity contact regions may be slightly overestimated in the theory of Persson at low squeezing pressure (see [7]), while the difference decreases at higher squeezing pressures and vanishes for complete contact. Here we will use the theory above to show that these two facts are in fact related.

Let us first derive an improved expression for  $f(\zeta)$  which occurs in (14). Let us write

$$f(\zeta) = \frac{\pi}{4} E^{*2} q_0 q^3 C(q) S(q). \quad (25)$$

Instead of (18) we then obtain

$$\int_{q_0}^q dq q^3 C(q) W(q) = \int_{q_0}^q dq q^3 C(q) P(q) S(q) \quad (26)$$

so that

$$S(q) = W(q)/P(q).$$

In [7] we have compared the theory prediction for the interfacial separation as a function of the squeezing pressure (which depends on  $U_{el}$  and hence on  $W(q)$ ) with MD calculations and found that for self-affine fractal surfaces with the fractal dimension  $D_f = 2.2$  good agreement with the simulation results are obtained if instead of  $W(q) = P(q)$  one uses

$$W(q) = P(q) [\beta + (1 - \beta) P^2(q)], \quad (27)$$

where  $\beta \approx 0.5$ , so that

$$S(q) = \beta + (1 - \beta) P^2(q). \quad (28)$$

This will result in

$$G(q) = \left( \frac{E^*}{\sigma_0} \right)^2 \frac{\pi}{4} \int_{q_0}^q dq q^3 C(q) S(q) \quad (29)$$

so that the area of real contact for a small load (where  $A \propto G^{-1/2}$  increases linear with the load) will be enhanced by a factor of  $1/\sqrt{\beta}$ . This is in agreement with the numerical simulation results [7–9].

I thank Carlos Campana, Martin Müser and Mark Robbins for a preprint of (an early version of) [2]. I also thank them for correspondence related to the topic of this study.

## References

- [1] Persson B N J 2007 *Phys. Rev. Lett.* **99** 125502
- [2] Campana C, Müser M H and Robbins M O 2008 *J. Phys.: Condens. Matter* submitted (Campana C, Müser M H and Robbins M O 2008 *Preprint* 0804.0062v2 [cond-mat.other])
- [3] Persson B N J 2001 *J. Chem. Phys.* **115** 3840
- [4] Persson B N J 2002 *Eur. Phys. J. E* **8** 385
- [5] Greenwood J A and Williamson J B P 1966 *Proc. R. Soc. A* **295** 300
- [6] Bush A W, Gibson R D and Thomas T R 1975 *Wear* **35** 87
- [7] Yang C and Persson B N J 2008 *J. Phys.: Condens. Matter* **20** 215214
- [8] Hyun S, Pei L, Molinari J-F and Robbins M O 2004 *Phys. Rev. E* **70** 026117
- [9] Campana C and Müser M H 2007 *Europhys. Lett.* **77** 38005